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SYSTOLIC INEQUALITIES AND MASSEY PRODUCTS IN SIMPLY-CONNECTED MANIFOLDS

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ABSTRACT

We show that the existence of a nontrivial Massey product in the cohomology ring $H^*(X)$ imposes global constraints upon the Riemannian geometry of a manifold X. Namely, we exhibit a suitable systolic inequality, associated to such a product. This generalizes an inequality proved in collaboration with Y. Rudyak, in the case when X has unit Betti numbers, and realizes the next step in M. Gromov's program for obtaining geometric inequalities associated with nontrivial Massey products. The inequality is a volume lower bound, and depends on the metric via a suitable isoperimetric quotient. The proof relies upon W. Banaszczyk's upper bound for the successive minima of a pair of dual lattices. Such an upper bound is applied to the integral lattices in homology and cohomology of X. The possibility of applying such upper bounds to obtain volume lower bounds was first exploited in joint work with V. Bangert. The latter work deduced systolic inequalities from nontrivial *cup-product* relations, whose role here is played by Massey products.

Contents

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1. Volume bounds and systolic category

A general framework for systolic geometry in a topological context was proposed in [KR06], in terms of a new numerical invariant called **systolic category**, denoted $\mathrm{cat}_{\mathrm{sys}}(X)$, of a space X. The terminology is inspired by the intriguing connection which emerges with the classical numerical invariant called the Lusternik-Schnirelmann category. Thus, we have the following.

- the two categories (i.e. the two integers) coincide for 2-complexes [KRS06];
- the two categories coincide for 3-manifolds, orientable or not [KR06, KR07];
- systolic category is a lower bound for the LS category for orientable 4-manifolds [DKR07];
- the two categories attain their maximal value simultaneously [KR06];
- both categories admit a lower bound in terms of real cup-length [Gr83, BK03];
- both categories are sensitive to Massey products [KL05, KR06, KR07, DKR07].

Definition 1.1: The stable k-systole of a Riemannian manifold is the least stable norm of a nonzero element in the integer lattice in its k-dimensional homology group with real coefficients.

A more detailed definition appears below, cf. formula (4.5).

The invariant $\mathrm{cat}_{\mathrm{sys}}$ is defined in terms of the existence of volume lower bounds of a certain type. Namely, these are bounds by products of lower-dimensional systoles. The invariant $\mathrm{cat}_{\mathrm{sys}}$ is, roughly, the greatest length d of a product

$$\prod_{i=1}^{d} \mathrm{sys}_{k_i}$$

of systoles which provides a universal lower bound for the volume, i.e. a curvature-independent lower bound of the following form:

$$\prod_{i=1}^{d} \mathrm{sys}_{k_i}(\mathcal{G}) \leq C\mathrm{vol}(\mathcal{G}),$$

see [KR06] for details. The definitions of the systolic invariants involved may also be found in [Gr83, CrK03, KL05].

We study stable systolic inequalities satisfied by an arbitrary metric $\mathcal{G}$ on a closed, smooth manifold X. We aim to go beyond the multiplicative structure, defined by the cup product, in the cohomology ring, whose systolic effects were studied in [Gr83, He86, BK03, BK04], and explore the systolic influence of Massey products.

Remark 1.2: This line of investigation is inspired by M. Gromov's remarks [Gr83, 7.4.C', p. 96] and [Gr83, 7.5.C, p. 102], outlining a program for obtaining geometric inequalities associated to nontrivial Massey products of any length. The first step in the program was carried out in [KR06] in the presence of a nontrivial triple Massey product in a manifold with unit Betti numbers.

In the present work, we exploit W. Banaszczyk's bound (4.4) for the successive minima of a pair of dual lattices, applied to the integral lattices in homology and cohomology of X. The possibility of exploiting such bounds to obtain inequalities was first demonstrated in the joint work with V. Bangert [BK03] on systolic inequalities associated to nontrivial cup product relations in the cohomology ring of X.

Whenever a manifold admits a nontrivial Massey product, we seek to exhibit a corresponding inequality for the stable systoles. While nontrivial cup product relations in cohomology entail stable systolic inequalities which are metric-independent and curvature-free [BK03], the influence of Massey products on systoles is more difficult to pin down. The inequalities obtained so far do depend mildly on the metric, via isoperimetric quotients, cf. (2.2).

The idea is to show that if, in a certain dimension $k \leq n$, one can span the cohomology by classes which can be expressed in terms of lower-dimensional classes by either Massey or cup products, then the stable k-systole (cf. Definition 4.6) admits a bound from below in terms of lower-dimensional stable systoles, and of certain isoperimetric constants of the metric, but no further metric data. Typical examples are inequalities (3.1), (3.2), (3.3).

Massey products and isoperimetric quotients are reviewed in Section 2. The theorems are stated in Section 3. Banaszczyk's results are reviewed in Section 4. The key notion of quasiorthogonal element of a Massey product is defined in Section 5. The theorems are proved in Section 6.

The basic reference for this material is M. Gromov's monograph [Gr99], with additional details in the earlier works [Gr83, Gr96]. For a survey of progress in systolic geometry up to 2003, see [CrK03]. More recent results include a study of optimal inequalities of Loewner type [Am04, IK04, BCIK07, KL05, KS06a], as well as near-optimal asymptotic bounds [BabB05, Ka03, KS05, KS06b, Sa04, Sa06, KSV05], while generalisations of Pu's inequality are studied in [BCIK05] and [BKSS06]. For an overview of systolic questions, see [Ka07].

2. Massey products and isoperimetric quotients

In Theorem 3.1, we will use a hypothesis which in the case of no indeterminacy of Massey products, amounts simply to requiring every cohomology class to be a sum of Massey products. In general, the condition is slightly stronger, and informally can be described by saying that any system of representatives of Massey products already spans the entire cohomology space.

Following the notation of [KR06], consider (homogeneous) cohomology classes u, v, w with $uv = 0 = vw$. Then the triple Massey product

$$\langle u, v, w \rangle \subset H^*_{\mathrm{dR}}$$

is defined as follows. Let a, b, c be closed differential forms whose homology classes are u, v, w respectively. Then $dx = ab$, $dy = bc$ for suitable differential forms x, y. Then $\langle u, v, w \rangle$ is defined to be the set of elements of the form

$$xc - (-1)^{|u|} ay,$$

see [Ma69, RT00] for more details. The set $\langle u, v, w \rangle$ is a coset with respect to the so-called **indeterminacy subgroup** $\mathrm{Indet} \subset H^{|u|+|v|+|w|-1}$, defined as follows:

$$\mathrm{Indet} = uH^{|v|+|w|-1} + H^{|u|+|v|-1}w. \tag{2.1}$$

A Massey product is said to be *nontrivial* if it does not contain 0.

Definition 2.1: Let $m \geq 1$. The $(3m-1)$-dimensional de Rham cohomology space of a manifold X is **of Massey type** if it has the following property.

Let $V \subset H^{3m-1}_{\mathrm{dR}}(X)$ be a subspace with nonempty intersection with every nontrivial triple Massey product $\langle u, v, w\rangle$, $u, v, w \in H^m_{\mathrm{dR}}(X)$. Then $V = H^{3m-1}_{\mathrm{dR}}(X)$.

Given a compact Riemannian manifold $(X, \mathcal{G})$, and a positive integer $k \leq \dim X$, denote by $\mathrm{IQ}_k = \mathrm{IQ}_k(\mathcal{G})$ the isoperimetric quotient, defined by

$$\mathrm{IQ}_k(\mathcal{G}) = \sup_{\alpha \in \Omega^k(X)} \inf_{\beta} \left\{ \tfrac{\|\beta\|^*}{\|\alpha\|^*} \,\middle|\, d\beta = \alpha \right\}, \tag{2.2}$$

where $\| \; \|^*$ is the comass norm [Fe74], and the supremum is taken over all exact k-forms. The relation of such quotients to filling inequalities is described in [Si05, Section 4, Proposition 1], cf. [Fe74, item 4.13].

3. The results

The following theorem generalizes [KR06, Theorem 13.1] to the case of arbitrary Betti number.

THEOREM 3.1: *Let X be a connected closed orientable smooth manifold. Let $m \geq 1$, and assume $b = b_m(X) > 0$. Furthermore, assume that the following three hypotheses are satisfied:*

1. *the cup product map $\cup\colon H^m_{\mathrm{dR}}(X) \otimes H^m_{\mathrm{dR}}(X) \to H^{2m}_{\mathrm{dR}}(X)$ is the zero map;*
2. *the space $H^{3m-1}_{\mathrm{dR}}(X)$ is of Massey type in the sense of Definition 2.1;*
3. *the group $H^{2m}(X, \mathbb{Z})$ is torsionfree.*

Then every metric $\mathcal{G}$ on X satisfies the inequality

$$\mathrm{stsys}_m(\mathcal{G})^3 \leq C(m)(b(1 + \log b))^3 \, \mathrm{IQ}_{2m}(\mathcal{G}) \, \mathrm{stsys}_{3m-1}(\mathcal{G}), \tag{3.1}$$

where $C(m)$ is a constant depending only on m.

Note that the dimensionality of the factor $\mathrm{IQ}_k(\mathcal{G})$ is $(\mathrm{length})^{+1}$, making inequality (3.1) scale-invariant, cf. [Gr83, 7.4.C', p. 96 and 7.5.C, p. 102].

The proof of Theorem 3.1 appears in Section 6.

An important special case is a lower bound for the total volume. While Hypothesis 2 of Theorem 3.1 is rather restrictive, similar inequalities can be proved in the presence of a nontrivial Massey product, even if Hypothesis 2 is not satisfied, provided one replaces the systole in the right hand side by the total volume. The simplest example of a theorem along these lines is the following.

Theorem 3.2: *Let X be a closed orientable smooth manifold of dimension 7. Assume that the following three hypotheses are satisfied:*

(1) *the cup product vanishes on $H^2_{\mathrm{dR}}(X)$;*
(2) *there are classes $u, v, w \in H^2_{\mathrm{dR}}(X)$ such that the triple Massey product $\langle u, v, w\rangle \subset H^5_{\mathrm{dR}}(X)$ is nontrivial;*
(3) *the group $H^4(X, \mathbb{Z})$ is torsionfree.*

Then every metric $\mathcal{G}$ on X satisfies the inequality

$$\mathrm{stsys}_2(\mathcal{G})^4 \le C(b_2(X))\ \mathrm{IQ}_4(\mathcal{G})\mathrm{vol}_7(\mathcal{G}), \tag{3.2}$$

where the constant $C(b_2(X)) > 0$ depends only on the second Betti number of X.

Examples of manifolds to which Theorem 3.1 and Theorem 3.2 can be applied, were constructed by A. Dranishnikov and Y. Rudyak [DR03].

Our Theorem 3.2 implies the following bound for the IQ-modified systolic category, cf. [KR06, Remark 13.1].

Corollary 3.3: *Under the hypotheses of Theorem 3.2, the manifold X satisfies the bound $\mathrm{cat}^{\mathrm{IQ}}_{\mathrm{sys}}(X) \ge 3$.*

Corollary 3.4: *Suppose in addition to the hypotheses of Theorem 3.2 that X is simply connected. Then $\mathrm{cat}^{\mathrm{IQ}}_{\mathrm{sys}}(X) \ge \mathrm{cat}_{\mathrm{LS}}(X)$.*

Proof. By [CLOT03, Theorem 1.50], the Lusternik–Schnirelmann category of X equals 3. ∎

Our last result attempts to go beyond both Theorem 3.1 and Theorem 3.2, in the sense of obtaining a lower bound for a k-systole other than the total volume, in a situation where Massey products do not necessarily span k-dimensional cohomology.

Proposition 3.5: *Consider a closed manifold X with a nontrivial triple Massey product containing an element $u \in H^5(X)$. Assume that the following three hypotheses are satisfied:*

(1) *the cup product vanishes on $H^2(X)$;*
(2) *the 8-dimensional cohomology of X is spanned by classes of type $u \cup v$ and w, where $v \in H^3(X)$, while $w \in H^8(X)$ is the cup square of a 4-dimensional class;*

(3) *the group* $H^4(X,\mathbb{Z})$ *is torsionfree.*

Then every metric $\mathcal{G}$ *on* X *satisfies the inequality*

$$\min\left\{\frac{\operatorname{stsys}_2(\mathcal{G})^3 \operatorname{stsys}_3(\mathcal{G})}{\operatorname{IQ}_4(\mathcal{G})},\ \operatorname{stsys}_4(\mathcal{G})^2\right\} \le C(X)\operatorname{stsys}_8(\mathcal{G}), \tag{3.3}$$

where $C(X) > 0$ *is a constant depending only on the homotopy type of* X*.*

The proof appears in Section 6.

4. Banaszczyk's bound for the successive minima of a lattice

Let B be a finite-dimensional Banach space, equipped with a norm $\| \ \|$. Let $L \subset B$ be a lattice of maximal rank $\operatorname{rank}(L) = \dim(B)$. Let $b = \operatorname{rank}(L) = \dim(B)$.

Definition 4.1: For each $k = 1, 2, \ldots, b$, define the **k-th successive minimum** λ_k of the lattice L by setting

$$\lambda_k(L, \| \ \|) = \inf\left\{\lambda\in\mathbb{R} \ \middle|\ \begin{array}{c}\exists \text{ lin. indep. } v_1,\ldots,v_k\in L\\ \text{with } \|v_i\|\le\lambda, \quad i=1,\ldots,k\end{array}\right\}. \tag{4.1}$$

In particular, the "first" successive minimum, $\lambda_1(L, \| \ \|)$, is the least length of a nonzero element in L.

Definition 4.2: Denote the "last" successive minimum by

$$\Lambda(L, \| \ \|) = \lambda_b(L, \| \ \|). \tag{4.2}$$

Definition 4.3: A linearly independent family

$$\{v_i\}_{i=1,\ldots,b} \subset L$$

is called **quasiorthogonal** if $\|v_i\| = \lambda_i$ for all $i = 1, \ldots, b$.

Note that a quasiorthogonal family spans a lattice of finite index in L, but may in general not be an integral basis, a source of some of the complications of the successive minimum literature.

Dually, we have the Banach space $B^* = \operatorname{Hom}(B, \mathbb{R})$, with norm $\| \ \|^*$, and dual lattice $L^* \subset B^*$, with $\operatorname{rank}(L^*) = \operatorname{rank}(L)$.

THEOREM 4.4 (W. Banaszczyk): *Every lattice L in every Banach space $(B, \| \, \|)$ satisfies the inequality*

$$\lambda_1(L, \| \, \|)\, \Lambda(L^*, \| \, \|^*) \leq Cb(1 + \log b), \tag{4.3}$$

for a suitable numerical constant C, where $b = \mathrm{rank}(L)$.

In fact, the upper bound is valid more generally for the product

$$\lambda_i(L)\lambda_{b-i+1}(L^*), \tag{4.4}$$

for all $i = 1, \ldots, b$ [Ban96].

Remark 4.5: A lattice $L \subset \mathbb{R}^b$ admits an **orthogonal** basis if and only if $\lambda_i(L)\lambda_{b-i+1}(L^*) = 1$ for all i. Thus, Banaszczyk's bound can be thought of as a measure of the quasiorthogonality of a lattice in Banach space.

Given a class $\alpha \in H_k(M; \mathbb{Z})$ of infinite order, we define the stable norm $\|\alpha_{\mathbb{R}}\|$ by setting

$$\|\alpha_{\mathbb{R}}\| = \lim_{m \to \infty} m^{-1} \inf_{\alpha(m)} \mathrm{vol}_k(\alpha(m)),$$

where $\alpha_{\mathbb{R}}$ denotes the image of α in real homology, while $\alpha(m)$ runs over all Lipschitz cycles with integral coefficients representing the multiple class $m\alpha$. The stable norm is dual to the comass norm $\| \, \|^*$ in cohomology, cf. [Fe74, BK03].

Definition 4.6: The **stable homology k-systole** of $(X, \mathcal{G})$ is

$$\mathrm{stsys}_k(\mathcal{G}) = \lambda_1(H_k(X, \mathbb{Z})_{\mathbb{R}}, \| \, \|), \tag{4.5}$$

where $\| \, \|$ is the stable norm.

5. Linearity vs. indeterminacy of triple Massey products

We will denote by $H^k_{\mathrm{dR}}(X, \mathbb{Z})$ the image of integral cohomology in real cohomology under inclusion of coefficients. Let $\{[v_i]\} \subset H^m_{\mathrm{dR}}(X, \mathbb{Z})$ be a quasiorthogonal family in the sense of Definition 4.3, with

$$\|v_i\|^* = \lambda_i(H^m_{\mathrm{dR}}(X, \mathbb{Z}), \| \, \|^*),$$

as in formula (4.1), where $\| \, \|^*$ is the comass norm. Here we assume, to simplify the calculations, that each m-form v_i minimizes the comass norm in its cohomology class. Given an exact $(2m)$-form $v_i \wedge v_j$, let w_{ij} be a primitive of least comass, cf. (2.2).

Definition 5.1: An element of the form

$$[w_{ij} \wedge v_k - (-1)^m v_i \wedge w_{jk}] \in \langle v_i, v_j, v_k \rangle$$

is called a **quasiorthogonal element of the Massey product** $\langle v_i, v_j, v_k \rangle$.

LEMMA 5.2: *Under the hypotheses of Theorem 3.1, the existence of a nontrivial Massey product implies the existence of a nonzero quasiorthogonal element of a suitable Massey product.*

Proof. The lemma follows by linearity, cf. (5.5). Since the detailed proof contains a delicate point involving indeterminacy, we include it here. By triviality of cup product hypothesis (1) of Theorem 3.1, for each pair of indices $1 \leq i, j \leq b_m(X)$, there is a $(2m-1)$-form w_{ij} solving the equation

$$v_i \wedge v_j = dw_{ij}. \tag{5.1}$$

Furthermore, given a metric $\mathcal{G}$, we can assume that w_{ij} satisfies the inequality

$$\|w_{ij}\|^* \leq \mathrm{IQ}_{2m}(\mathcal{G}) \|v_i \wedge v_j\|^*, \tag{5.2}$$

cf. formula (2.2).

Using index notation (Einstein summation convention), let i, j, k run from 1 to $b_m(X)$. Let $\langle u, v, w \rangle$ be a nontrivial Massey product, as in Theorem 3.1. Choose representative differential forms $\alpha = \alpha^i v_i \in u$, $\beta = \beta^j v_j \in v$, and $\gamma = \gamma^k v_k \in w$. Then

$$\alpha \wedge \beta = \left(\alpha^i v_i\right) \wedge \left(\beta^j v_j\right) = \alpha^i \beta^j v_i \wedge v_j = \alpha^i \beta^j dw_{ij} = d\left(\alpha^i \beta^j w_{ij}\right), \tag{5.3}$$

and similarly $\beta \wedge \gamma = d\left(\beta^j \gamma^k w_{jk}\right)$. Since the Massey product is nontrivial, we obtain a nonzero cohomology class

$$\left[\alpha^i \beta^j w_{ij} \wedge \gamma - (-1)^m \alpha \wedge \beta^j \gamma^k w_{jk}\right] \neq 0 \in H^{3m-1}_{\mathrm{dR}}(X). \tag{5.4}$$

By linearity, we have

$$\alpha^i \beta^j w_{ij} \wedge \gamma - (-1)^m \alpha \wedge \beta^j \gamma^k w_{jk} = \alpha^i \beta^j \gamma^k \left(w_{ij} \wedge v_k - (-1)^m v_i \wedge w_{jk}\right). \tag{5.5}$$

Therefore

$$\alpha^i \beta^j \gamma^k \left[w_{ij} \wedge v_k - (-1)^m v_i \wedge w_{jk}\right] \neq 0 \in H^{3m-1}_{\mathrm{dR}}(X). \tag{5.6}$$

In fact, the nontriviality of the Massey product yields the stronger conclusion that we have a nonzero class in the quotient

$$\alpha^i\beta^j\gamma^k\,[w_{ij}\wedge v_k-(-1)^m v_i\wedge w_{jk}]\neq 0\in H^{3m-1}_{\rm dR}(X)/\,\mathrm{Indet}, \tag{5.7}$$

cf. (2.1). Hence, for suitable indices $1\le s,t,r\le b_m(X)$, we obtain a nonzero class

$$[w_{st}\wedge v_r-(-1)^m v_s\wedge w_{tr}]\in\langle v_s,v_t,v_r\rangle$$

in $H^{3m-1}_{\rm dR}(X)/\,\mathrm{Indet}$. Note that this conclusion differs from the assertion that the Massey product $\langle v_s,v_t,v_r\rangle$ is nontrivial, since its indeterminacy subspace may be different from that of the Massey product $\langle u,v,w\rangle$. ∎

Remark 5.3: The indices s,t,r above may depend on the various choices involved in the construction, but the key estimate (6.2) remains valid, due to the uniqueness of the least natural number, by the well-ordered property of $\mathbb{N}$.

LEMMA 5.4: *Let $x_0\in H_{3m-1}(X,\mathbb{R})$ be a fixed nonzero class. The hypotheses of Theorem 3.1 imply the existence of a nonzero quasiorthogonal element of a Massey product, which pairs nontrivially with x_0.*

Proof. Consider the family of all quasiorthogonal elements q_i of Massey products. Let V be the vector space spanned by all such elements q_i. By (5.5), the space V meets every nontrivial Massey product. By our Massey-type hypothesis, we have

$$V=H^{3m-1}_{\rm dR}(X). \tag{5.8}$$

Choose any cohomology class a which pairs nontrivially with x_0, i.e. $a(x_0)\neq 0$. By (5.8), we can write $a=a^iq_i$, where q_i are quasiorthogonal elements of Massey products. Thus $a^iq_i(x_0)\neq 0$ and by linearity, one of the quasiorthogonal elements, say q_{i_0}, also pairs nontrivially with x_0. ∎

LEMMA 5.5: *Assume $H^{2m}(X,\mathbb{Z})$ is torsionfree. Then every quasiorthogonal element of a Massey product satisfies the integrality condition*

$$\int_{x_0}\langle v_s,v_t,v_r\rangle\in\mathbb{Z}, \tag{5.9}$$

where $x_0\in H_m(X,\mathbb{Z})$ is any integral class.

Proof. Choose representatives for the v_i in the cohomology group with integer coefficients $H^m(X,\mathbb{Z})$ in the sense of singular cohomology theory. We denote these representatives $\tilde{v}_i$. Choose an m-cocycle $\tilde{\tilde{v}}_i \in \tilde{v}_i$. Note that the class

$$[\tilde{\tilde{v}}_s \wedge \tilde{\tilde{v}}_t] \in H^{2m}(X, \mathbb{Z})$$

vanishes integrally, and thus the Massey product $\langle \tilde{v}_s, \tilde{v}_t, \tilde{v}_r \rangle$ is defined over $\mathbb{Z}$. The lemma now follows from the compatibility of the de Rham and the singular Massey product theories, verified in [Ma69] and [KR06, Section 11], in terms of differential graded associative (dga) algebras, cf. Remark 5.6 below. ∎

Remark 5.6: The following three remarks were kindly provided by R. Hain (see [KR06, Ka07] for more details).

1. If A^* and B^* are dga algebras (not necessarily commutative) and $f : A^* \to B^*$ is a dga homomorphism that induces an isomorphism on homology, then Massey products in $H^*(A^*)$ and $H^*(B^*)$ correspond under $f^* : H^*(A^*) \to H^*(B^*)$.

2. If M is a manifold, then there is a dga K^* that contains both the de Rham complex $A^*(M)$ of M, and also the singular cochain complex $S^*(M)$ of M. The two inclusions

$$A^*(M) \to K^* \leftarrow S^*(M)$$

are both dga quasi-isomorphisms (i.e. induce isomorphism in cohomology), cf. [FHT98, Corollary 10.10].

3. The point is that the inclusions $A^*(M) \to K^* \leftarrow S^*(M)$ are both dga homomorphisms (and quasi-isomorphisms), even though $A^*(M)$ is commutative and $S^*(M)$ is not. Combining these two remarks, we see that Massey products in singular cohomology and in de Rham cohomology correspond. The complex K^* is a standard tool in rational homotopy theory. It is defined as follows. Let Simp be the simplicial set of smooth singular simplices of M. Then K^* is Thom-Whitney complex of differential forms on Simp.

6. Proofs of main results

Proof of Theorem 3.1. Let $\mathcal{G}$ be a metric on X. Let $\| \; \|$ be the associated stable norm in homology. Choose a class $x_0 \in H_{3m-1}(X, \mathbb{Z})_{\mathbb{R}}$ satisfying

$$\|x_0\| = \operatorname{stsys}_{3m-1}(X, \mathcal{G}) = \lambda_1(H_{3m-1}(X, \mathbb{Z})_{\mathbb{R}}, \| \; \|). \tag{6.1}$$

We can then choose a cohomology class $\alpha \in H^{3m-1}_{\mathrm{dR}}(X, \mathbb{Z})$ which pairs nontrivially with the class x_0, i.e. satisfying $\alpha(x_0) \neq 0$. We will write this condition suggestively as $\int_{x_0} \alpha \neq 0$. A reader familiar with normal currents can interpret integration in the sense of the minimizing normal current representing the class x_0. Otherwise, choose a rational Lipschitz m-cycle of volume ϵ-close to the value (6.1), and let ϵ tend to zero at the end of the calculation below.

By Lemma 5.4, the class α can be replaced by a quasiorthogonal element of a Massey product $\langle v_s, v_t, v_r\rangle$, which also pairs nontrivially with x_0.

Recall that $\| \, \|^*$ is the comass norm in cohomology. Changing orientations if necessary, we obtain from (5.9) that

$$1 \leq \int_{x_0} w_{st} \wedge v_r - (-1)^m v_s \wedge w_{tr}, \tag{6.2}$$

and therefore

$$1 \leq C(m)\left(\|w_{st}\|^* \|v_r\|^* + \|v_s\|^* \|w_{tr}\|^*\right)\|x_0\|, \tag{6.3}$$

where $C(m)$ depends only on m. Now by (5.2), we have

$$\begin{aligned} 1 &\leq 2C(m)\|v_s\|^*\|v_t\|^*\|v_r\|^* \,\mathrm{IQ}_{2m}(\mathcal{G})\|x_0\| \\ &= 2C(m)\lambda_s\lambda_t\lambda_r \,\mathrm{IQ}_{2m}(\mathcal{G})\|x_0\| \\ &\leq 2C(m)\left(\Lambda\left(H^m_{\mathrm{dR}}(X,\mathbb{Z}), \|\,\|^*\right)\right)^3 \mathrm{IQ}_{2m}(\mathcal{G})\|x_0\|, \end{aligned}$$

by Definition 4.2 of the "last" successive minimum $\Lambda(L)$. Finally, by definition we have $\mathrm{stsys}_m(\mathcal{G}) = \lambda_1(H_m(X), \|\,\|)$, where $\|\,\|$ is the stable norm, and therefore

$$\mathrm{stsys}_m(\mathcal{G})^3 \leq 2C(m)\left(\lambda_1\left(H_m(X)\right)\Lambda(H^m(X))\right)^3 \mathrm{IQ}_{2m}(\mathcal{G})\|x_0\|. \tag{6.4}$$

Applying Banaszczyk's inequality (4.3), we obtain

$$\begin{aligned} \mathrm{stsys}_m(\mathcal{G})^3 &\leq C(m)(b(1+\log b))^3 \,\mathrm{IQ}_{2m}(\mathcal{G})\|x_0\| \\ &= C(m)(b(1+\log b))^3 \,\mathrm{IQ}_{2m}(\mathcal{G})\,\mathrm{stsys}_{3m-1}(\mathcal{G}), \end{aligned}$$

where $b = b_m(X)$, while the new coefficient $C(m)$ incorporates the numerical constant from Banaszczyk's inequality. This completes the proof of Theorem 3.1. ∎

Proof of Theorem 3.2. Exploiting the orientability of X, we represent its fundamental cohomology class as a product $\langle u_1, u_2, u_3\rangle \cup u_4$, with $u_i \in H^2(X)$. Here we write $\langle u_1, u_2, u_3\rangle$ as shorthand for an orthogonal element of a Massey product, while u_4 may be chosen to be any class which pairs nontrivially with the Poincaré dual of $\langle u_1, u_2, u_3 \rangle$. Relation (5.9) is replaced by the following integrality relation among the elements $v_i \in H^2_{\rm dR}(X)$ of a quasiorthogonal family:

$$\int_X \langle v_s, v_t, v_r \rangle \cup v_p \in \mathbb{Z} \setminus \{0\}. \tag{6.5}$$

The rest of the proof is similar. ∎

Proof of Proposition 3.5. Choose a class $x_0 \in H_8(X, \mathbb{Z})_{\mathbb{R}}$ satisfying $\|x_0\| = \lambda_1(H_8(X, \mathbb{Z})_{\mathbb{R}}; \| \, \|)$. The class x_0 pairs nontrivially with one of the classes $u \cup v$ or w.

If for some Massey product u, we have $\int_{x_0} u \cup v \neq 0$, we argue as in the proof of Theorem 3.1, exploiting the hypothesis that the cup product in $H^2(X)$ is trivial, in order to define the quasiorthogonal elements of triple Massey products.

If the class w satisfies $w(x_0) \neq 0$, we argue with a quasiorthogonal family in $H^4_{\rm dR}(X, \mathbb{Z})$ as in [BK03] to obtain the lower bound for the stable norm of x_0 in terms of $\text{stsys}_4(\mathcal{G})^2$. ∎

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References

[Am04] B. Ammann, *Dirac eigenvalue estimates on two-tori,* Journal of Geometry and Physics **51** (2004), 372–386.

[BabB05] I. Babenko, F. Balacheff, *Géométrie systolique des sommes connexes et des revêtements cycliques,* Mathematische Annalen **333** (2005), 157–180.

[Ban93] W. Banaszczyk, *New bounds in some transference theorems in the geometry of numbers,* Mathematische Annalen **296** (1993), 625–635.

[Ban96] W. Banaszczyk, *Inequalities for convex bodies and polar reciprocal lattices in* $\mathbb{R}^n$*. II. Application of K-convexity,* Discrete & Computational Geometry **16** (1996), 305–311.

[BCIK05] V. Bangert, C. Croke, S. Ivanov and M. Katz, *Filling area conjecture and ovalless real hyperelliptic surfaces,* Geometric and Functional Analysis (GAFA) **15** (2005), 577–597. See `arXiv:math.DG/0405583`

[BCIK07] V. Bangert, C. Croke, S. Ivanov and M. Katz, *Boundary case of equality in optimal Loewner-type inequalities.* Transactions of the American Mathematical Society **359** (2007), 1–17. See `arXiv:math.DG/0406008`

[BK03] V. Bangert and M. Katz, *Stable systolic inequalities and cohomology products,* Communications on Pure and Applied Mathematics **56** (2003), 979–997. see `arXiv:math.DG/0204181`

[BK04] V. Bangert and M. Katz, *An optimal Loewner-type systolic inequality and harmonic one-forms of constant norm,* Communications in Analysis and Geometry **12** (2004), 703–732. See `arXiv:math.DG/0304494`

[BKSS06] V. Bangert, M. Katz, S. Shnider and S. Weinberger, E_7, *Wirtinger inequalities, Cayley 4-form, and homotopy,* Duke Mathematical Journal, accepted

[CLOT03] O. Cornea, G. Lupton, J. Oprea and D. Tanré, *Lusternik-Schnirelmann category,* Mathematical Surveys and Monographs, 103, American Mathematical Society, Providence, RI, 2003.

[CrK03] C. Croke and M. Katz, *Universal volume bounds in Riemannian manifolds,* Surveys in Differential Geometry **8** (2003), 109–137. see `arXiv:math.DG/0302248`

[DKR07] A. Dranishnikov, M. Katz and Y. Rudyak, *Small values of Lusternik-Schnirelmann and systolic categories for manifolds.* See arXiv:0706.1625

[DR03] A. Dranishnikov and Y. Rudyak, *Examples of non-formal closed* $(k-1)$*-connected manifolds of dimensions* $\geq 4k-1$, Proceedings of the American Mathematical Society **133** (2005), 1557–1561. See `arXiv:math.AT/0306299`

[Fe74] H. Federer, *Real flat chains, cochains, and variational problems,* Indiana University Mathematics Journal **24** (1974), 351–407.

[FHT98] Y. Felix, S. Halperin and J.-C. Thomas, sl Rational Homotopy Theory, Graduate Texts in Mathematics, 205, Springer-Verlag, New York, 2001.

[Gr83] M. Gromov, *Filling Riemannian manifolds,* Journal of Differential Geometry **18** (1983), 1–147.

[Gr96] M. Gromov, *Systoles and intersystolic inequalities,* in *Actes de la Table Ronde de Géométrie Différentielle* (Luminy, 1992), Séminaires et Congrès, **1**, Soc. Math. France, Paris, 1996, pp. 291–362.
`www.emis.de/journals/SC/1996/1/ps/smf_sem-cong_1_291-362.ps.gz`

[Gr99] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces,* Progress in Mathematics, 152, Birkhäuser, Boston, 1999.

[He86] J. Hebda, *The collars of a Riemannian manifold and stable isosystolic inequalities,* Pacific Journal of Mathematics **121** (1986), 339–356.

[IK04] S. Ivanov and M. Katz, *Generalized degree and optimal Loewner-type inequalities,* Israel Journal of Mathematics **141** (2004), 221–233. See `arXiv:math.DG/0405019`

[Ka03] M. Katz, *Four-manifold systoles and surjectivity of period map,* Commentarii Mathematici Helvetici **78** (2003), 772–786. See `arXiv:math.DG/0302306`

[Ka07] M. Katz, *Systolic geometry and topology,* With an appendix by Jake P. Solomon, Mathematical Surveys and Monographs **137**, American Mathematical Society, Providence, RI, 2007.

[KL05] M. Katz and C. Lescop, *Filling area conjecture, optimal systolic inequalities, and the fiber class in abelian covers. Geometry, spectral theory, groups, and dynamics,* in *Contemporary Mathematics*, 387, American Mathematical Society Providence, RI, 2005, pp. 181–200. See `arXiv:math.DG/0412011`

[KR06] M. Katz and Y. Rudyak, *Lusternik-Schnirelmann category and systolic category of low dimensional manifolds*, Communications on Pure and Applied Mathematics **59** (2006), 1433–1456. Available at the site `arXiv:math.DG/0410456`

[KR07] M. Katz and Y. Rudyak, *Bounding volume by systoles of 3-manifolds*, See `arXiv:math.DG/0504008`

[KRS06] M. Katz, Y. Rudyak and S. Sabourau, *Systoles of* 2*-complexes, Reeb graph, and Grushko decomposition,* International Math. Research Notices **2006** (2006). Art. ID 54936, pp. 1–30. See `arXiv:math.DG/0602009`

[KS05] M. Katz and S. Sabourau, *Entropy of systolically extremal surfaces and asymptotic bounds*, Ergodic Theory and Dynamical Systems **25** (2005), 1209–1220. See `arXiv:math.DG/0410312`

[KS06a] M. Katz and S. Sabourau, *Hyperelliptic surfaces are Loewner*, Proceedings of the American Mathematical Society **134** (2006), 1189–1195. See `arXiv:math.DG/0407009`

[KS06b] M. Katz and S. Sabourau, *An optimal systolic inequality for CAT(0) metrics in genus two,* Pacific Journal of Mathematics **227** (2006), 95–107. See `arXiv:math.DG/0501017`

[KSV05] M. Katz, M. Schaps and U. Vishne, *Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups,* Journal of Differential Geometry **76** (2007), 399–422. See `arXiv:math.DG/0505007`

[Ma69] J. May, *Matric Massey products,* Journal of Algebra **12** (1969) 533–568.

[RT00] Y. Rudyak and A. Tralle, *On Thom spaces, Massey products, and nonformal symplectic manifolds,* International Mathenatics Resesearch Notices **10** (2000), 495–513.

[Sa04] S. Sabourau, *Systoles des surfaces plates singulières de genre deux,* Math. Zeitschrift **247** (2004), 693–709.

[Sa06] S. Sabourau, *Systolic volume and minimal entropy of aspherical manifolds,* Journal of Differential Geometry **74** (2006), 155–176. See `arXiv:math.DG/0603695`

[Si05] J. Sikorav, *Bounds on primitives of differential forms and cofilling inequalities,* See `arXiv:math.DG/0501089`